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Spatiotemporal and Statistical Symmetries

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The notion of symmetries, either statistical or deterministic, can be useful for the characterization of complex systems and their bifurcations. In this paper, we investigate the connection between the (microscopic) spatiotemporal symmetries of a space-time function $u(x, t)$, on the one hand, and the (macroscopic) symmetries of statistical quantities such as the spatial (resp. temporal) two-point correlations and the spatial (resp. temporal) average, on the other hand. We show, how, under certain conditions, these symmetries are related to the symmetries of the orbits described by $u(x, t)$ in the characteristic (phase) spaces. We also determine the largest group of spatiotemporal symmetries (in the sense introduced in our earlier work) satisfied by a given space-time function $u(x, t)$ and indicate how to extract the subgroups of point symmetries, namely those directly implemented on the space and time variables. Conversely, we determine all the functions invariant by a given space-time symmetry group. Finally, we illustrate all the previous points with specific examples.

KEY WORDS: Spatiotemporal complexity; spatiotemporal symmetries; statistical symmetries, two-point correlations; biorthogonal decomposition.

1. INTRODUCTION

Recently, a number of studies have concentrated on exploring spatially extended dynamical systems displaying spatiotemporal complex behavior (see ref. 2 and references therein). Such behavior, ubiquitous in nature, has been observed in many laboratory experiments[(2)], and in models such as coupled map lattices (CMLs) and numerical solutions of partial differential

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equations (PDEs). Due to the complexity of these systems, only a few rigorous results on the nature of the dynamics have been obtained (see ref. 3 for PDEs in infinitely extended domains and ref. 4 for CMLs). Although it is expected that future theoretical advances will be greatly enhanced by computational and experimental observations (see, e.g. refs. 5–11), one of the most challenging issues lies in the methods used to analyze complex numerical and experimental data, which have consisted, so far, mostly of statistical techniques.[9–11] The connection with space-time deterministic behavior is then unclear. An example of this issue is furnished by three recent experiments[12–14] which revealed symmetry properties of the time and/or space averages of individually asymmetric chaotic patterns. As in classical temporal dynamical systems, symmetries in spatiotemporal dynamical systems may play a fundamental role which deserves further investigation. One important issue, addressed in this paper, is the implication of statistical symmetries, namely the symmetries of the (macroscopic) statistics, for the symmetries of the (microscopic) space-time behavior $u(x, t)$, and vice versa.

The concept of spatial symmetries is classical in dynamical systems theory (see, e.g., ref. 15) and examples are numerous in physics, particularly in fluid mechanics. For example, at low Reynolds numbers, the wake flow behind a cylinder is steady, namely time independent, and invariant under reflection about its center plane $x_2 = 0$ [x_1 and x_3 denoting the streamwise and spanwise (along the cylinder axis) directions respectively, x_2 the remaining direction], so that

$$\forall x_1, x_2, x_3, \quad \forall t, \qquad u_i(x_1, x_2, x_3, t) = \varepsilon_i u_i(x_1, -x_2, x_3, t)$$

where

$$\begin{aligned} \varepsilon_i &= +1 \qquad \text{if} \quad i = 1 \text{ or } 3 \\ &= -1 \qquad \text{if} \quad i = 2 \end{aligned}$$

Here, u_i denotes the velocity component in the x_i-direction. As soon as the flow becomes unsteady, such spatial symmetry is *instantaneously* lost, but can be recovered in a *spatiotemporal* sense. Indeed, the (spatially and temporally) periodic Karman street, resulting from the primary wake flow instability, is invariant under the reflection symmetry about its center plane only after a time shift, equal to a half-period, $T/2$,

$$\forall x_1, x_2, x_3, \quad \forall t, \qquad u_i(x_1, x_2, x_3, t) = \varepsilon_i u_i(x_1, -x_2, x_3, t + T/2)$$

Similarly, as the street of vortices is convected downstream (without deformation if one neglects the wake decay due to viscous dissipation), its (streamwise) spatiotemporal dynamics is that of a traveling wave, so that, rather than staying invariant under a spatial shift at a given time, it is invariant under the simultaneous action of spatial and temporal shifts, namely

$$\forall x_1, x_2, x_3, \quad \forall t, \qquad \vec{u}(x_1, x_2, x_3, t) = \vec{u}(x_1 + x_0, x_2, x_3, t + t_0)$$
$$\forall x_0, t_0 \quad \text{such that} \quad x_0 + ct_0 = 0$$

c is the propagation speed of the wave. Such spatiotemporal symmetry is, of course, characteristic of all traveling waves. In the previous examples, it is interesting to note that the existence of a spatiotemporal symmetry implies that both the time average of the flow and its (spatial) two-point correlation are invariant under the spatial symmetry, while the flow itself, at all times, is not. These features are often observed even in more complex situations such as fully developed turbulence where it is well known that both the (time) mean flow and the Reynolds stresses are invariant under spatial symmetries while the instantaneous velocity fields are not. This manifests itself as the presence of coherent structures which are symmetric in an average sense only. This is the case, for instance, of streamwise vortices in the wall region of a turbulent boundary layer, which seldom appear in pairs of symmetric, counterrotating patterns in cross sections of the flow (see, e.g. refs. 16 and 17 and references therein). Nevertheless, most techniques used for extracting coherent structures from a turbulent background are based on averaging procedures and deduce symmetric vortices. As we recalled earlier, pattern on average has also been observed experimentally in (spatiotemporal) chaotic regimes in Rayleigh–Bénard convection[12] in the Faraday experiment,[13] and in rotating thermal convection[14]: the time average has the symmetry of the boundary conditions even though none of the time-instantaneous velocity fields has this symmetry. Whether the symmetry of the time average of the solution is a manifestation of the symmetry of the orbit in phase space or not is an open question. We show in this paper that a better connection is provided by the notion of space-time symmetry, which, in case of a temporal part directly implemented on the time variable, is equivalent to the symmetry of the orbit in phase space. In contrast, while the symmetry of the time average may be the manifestation of a spatiotemporal symmetry (since the latter implies the former), this is not necessarily the case.

In view of the physical examples given above, we concentrate in this paper on the important role played by spatiotemporal symmetries (introduced in ref. 1) and point out the difference and connection between those and the spatial and temporal symmetries of statistical quantities (particularly those of the averages and the two-point correlations). We then define and characterize all the spatiotemporal symmetries satisfied by any given function or signal $u(x, t)$. These symmetries form a group, the largest group of (finite and infinite) symmetries satisfied by $u(x, t)$, for which we give a method of identification. We then show that the existence of a spatiotemporal symmetry for which the temporal part is a point symmetry is a necessary and sufficient condition for the orbit in the spatial characteristic space to be spatially invariant and that the time average inherits this symmetry. The converse, however, is not true, namely one can find situations where the time average is symmetric without the orbit in phase space having this property. We also recall that the spatial two-point correlation satisfies such symmetry as well and, when this is the case, there exists a spatiotemporal symmetry.[18] All this, of course, remains valid when space and time are interchanged.

Thus, the method exposed in this paper, based on biorthogonal decompositions, provides a universal procedure for detecting all (spatiotemporal) symmetries defined in ref. 1. Indeed, the power of such decompositions (see Section 2) lies in the fact that all irreducible representations of the group of symmetries relevant in the analysis of $u(x, t)$ are naturally realized in the spatial and temporal characteristic spaces $\chi(X)$ and $\chi(T)$. Since the operator U [see Eq. (2.1)] realizes an equivalence (in the usual sense of the representations of groups) between the symmetries realized in $\chi(X)$ and those realized in $\chi(T)$, a description of the irreducible representations easily follows. The apparent simplicity of our proofs has its roots in the powerful theory of operators acting on Hilbert spaces and shows how such a framework is well adapted to the study of extended spatiotemporal dynamical systems and their symmetries, as it is in quantum mechanics.[19]

The paper is organized as follows. In Section 2, after recalling the spectral decomposition of the operator U (referred to as a biorthogonal decomposition), we point out that such decomposition differs from that of the correlation operators U^*U (or UU^*) (the asterisk denoting the adjoint operator). In particular, we list all the operators U_s, different from U, for which the correlation operators coincide. In Section 3, we recall the definition of a spatiotemporal symmetry of a function $u(x, t)$,[1] and define and extract the largest group of such spatiotemporal symmetries satisfied by $u(x, t)$. We then show that the existence of a spatiotemporal symmetry whose temporal component is a point symmetry is a necessary and sufficient condition for the orbit in the characteristic space $\chi(X)$ to be invariant under the spatial component of the symmetry. Moreover, the two-point correlations of $u(x, t)$ and the averages of $u(x, t)$ (the latter only under specific conditions on the nature of the spatiotemporal symmetry) both inherit the spatial or temporal components of such symmetries, but the converse, however, is not true. In Section 4, we determine all the functions $u(x, t)$ which are invariant under a given spatiotemporal symmetry group. Finally, in Section 5, we give examples which illustrate our methods and make the distinction between the various types of symmetries.

2. SPECTRAL DECOMPOSITIONS

2.1. The operator *U*. Biorthogonal Decompositions

Given a function $u(x, t)$, we consider a Hilbert space $H(X)$ of functions of $x \in X$ ($X \subset R$) and a Hilbert space $H(T)$ of functions of $t \in T$ ($T \subset R$) such that

$$\forall \varphi \in H(X), \qquad (U\varphi)(t) = \int_X u(x, t)\, \varphi(x)\, dm(x) \tag{2.1}$$

defines a linear operator from $H(X)$ to $H(T)$; $dm(x)$ denoting the measure defining the scalar product in $H(X)$. (For the sake of simplicity, we restrict our analysis in this paper to functions defined in one spatial dimension.) Similarly, $d\tilde{m}(t)$ denotes the measure defining the scalar product in $H(T)$, so that

$$(U^*\psi)(x) = \int_T \overline{u(x, t)}\, \psi(t)\, d\tilde{m}(t) \tag{2.1'}$$

where the bar denotes the complex conjugate, is the adjoint operator of U. Notice that, for a fixed choice of $H(X)$ and $H(T)$, e.g. $L^2(X)$ and $L^2(T)$, the possibility of defining (2.1) and (2.1′) as operators imposes some restrictions on $u(x, t)$. For instance, when $H(X)$ and $H(T)$ are identical to $L^2(X)$ and $L^2(T)$, respectively, $u(x, t)$ must be in $L^\infty(X, T)$ if one wants a bounded operator U. Conversely, for a given class of functions $u(x, t)$, it is clear that each appropriate choice of the spaces $H(X)$ and $H(T)$ that permits the definition of the (bounded or unbounded) operators U and U^* by (2.1) and (2.1′) may give rise to specific properties of the dynamics, particularly in terms of the symmetries (see ref. 18, where such properties were exploited). This issue is similar to the problem of selecting a space of functions for the solutions of a PDE, which is achieved by either mathematical or physical reasons. We then consider the spectral decomposition of the operator U, whose kernel is precisely the spatiotemporal dynamics of interest, $u(x, t)$, or equivalently, that of the operator V:

$$V = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \tag{2.2}$$

Such a decomposition has been referred to as "biorthogonal decomposition"[20, 18], (see also ref. 21 for its application to fluid mechanics), since it decomposes $u(x, t)$ into orthogonal modes of $H(X)$ and orthogonal modes of $H(T)$ with a one-to-one correspondence between both sets of modes. If the operator U is compact, then we can write

$$u(x, t) = \sum_n A_n \phi_n(x) \psi_n(t) \tag{2.3}$$

with $A_1 \geqslant A_2 \geqslant \cdots > 0$, and

$$(\phi_n, \phi_m)_{H(X)} = (\psi_n, \psi_m)_{H(T)} = \delta_{n,m}$$

which converges in norm. The parentheses denote the respective scalar products in $H(X)$ and $H(T)$. Hereafter, the spatial eigenmodes ϕ_n are called *topos* and the temporal eigenmodes ψ_n are called *chronos*. The one-to one correspondence between topos and chronos is given by the operator U itself:

$$U\phi_n = A_n \psi_n$$

or, equivalently, the operator U^*:

$$U^* \psi_n = A_n \phi_n$$

which we call "dispersion relation." The spatiotemporal dynamics can be studied as an orbit $\xi_t(x)$ defined by $\forall x \in X$, $\xi_t(x) = u(x, t)$ in the spatial characteristic space $\chi(X) = \mathrm{Ker}(U)^\perp$ spanned by the ϕ_n's or as an orbit $\eta_X(t)$ defined by $\forall t \in T$, $\eta_x(t) = u(x, t)$ in the temporal characteristic space $\chi(T) = \mathrm{Ker}(U^*)^\perp$ spanned by the ψ_n's, these two spaces sharing the same, minimal dimension.[20] The extension to noncompact operators with eventually a Carleman or generalized kernel $u(x, t)$ or a continuous component of the spectrum can be achieved by application of the theory of eigenfunction expansion of self-adjoint operators.[18] Such a decomposition has its origins in the spectral decomposition of operators with symmetric kernels, which can be found early in ref. 22 for compact operators and for operators with Carleman kernels in ref. 23. It has been proposed as a tool for studying spatiotemporal continuous and discrete dynamical systems[1, 20] and applied to the derivation of a space-time theory of fully developed turbulence,[18, 24] transition to turbulence,[20, 25] dispersive chaos and related states in binary fluid convection,[11, 26] wave propagation phenomena and their bifurcations,[27, 28] coupled map lattices,[20, 29] and solutions of the Kuramoto–Sivashinsky equation.[1, 30]

2.2. The Correlation Operator U^*U

The probability theory tool referred to as the Karhunen–Loève (KL) expansion or proper orthogonal decomposition,[31, 32] in which the sampling variable is time, corresponds to the spectral decomposition of U^*U whose kernel is the spatial two-point correlation. It has been proposed by Lumley for its application to turbulence[33, 34] (see also refs. 16 and 17) and applied in a number of studies (see the review, in refs. 35 and 36 and references therein). In our work, it is fundamental to consider the operator U or V for a *simultaneous* treatment of space and time. Moreover, in the previous statistical context, the introduction of averaging techniques different from the time average necessarily appearing in U^*U, such as an average over the spatial symmetries of the system considered, is not related to the issues raised in this paper.[1, 21] The latter technique, introduced in ref. 36, and used, for instance in ref. 37, to enlarge the statistical ensemble of data of the KL expansion, was proposed in ref. 38 to derive reduced dynamical systems which preserve the symmetry of the original PDE.

Returning to the main difference between the operator U^*U and the operator U, the former acts on the same Hilbert space, while the latter does not. To illustrate this point, we consider an operator acting on H, Q: $H \to H$, for which the eigenfunction problem is

$$Q\varphi = A\varphi \tag{2.4}$$

It is well known that in this case, if φ is an eigenvector of (2.4), then $c\varphi$, $c \in C$, is also an eigenvector. The situation is fundamentally different for the spectral analysis of (2.1), where the operator U acts from one Hilbert space to another one, i.e.,

$$U\colon H(X) \to H(T) \tag{2.5}$$

since the equation

$$U\varphi = A\psi \tag{2.6}$$

is then insufficient for determining the eigenvectors. We now need to solve *simultaneously* the two equations

$$\begin{aligned} U\varphi &= A\psi \\ U^*\psi &= A\varphi \end{aligned} \tag{2.7}$$

or, equivalently,

$$\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{2.8}$$

where the matrix operator is precisely the operator V introduced in (2.2). It then becomes clear that a problem for which the equations are given by (2.8) is not equivalent to the spectral analyses of U^*U and UU^* for which the eigenequations are

$$\begin{aligned} U^*U\varphi &= A^2\varphi \\ UU^*\psi &= A^2\psi \end{aligned} \tag{2.9}$$

Indeed a solution of (2.8) is a solution of (2.9), but the converse is not necessarily true. The dispersion relation $\phi \leftrightarrow \psi$ may be lost in (2.9), as becomes clear in the next subsection and in Examples 5.3, 5.4, and 5.6 given in Section 5.

2.3. Identification of All Functions $u(x, t)$ with the Same Correlations

Now, one can ask the following question: what are the spatiotemporal functions $u(x, t)$ with the same spatial and temporal two-point correlations $R(x, y)$ and $L(s, t)$, i.e.,

$$R(x, y) = \int_T u(x, t)\,\overline{u(y, t)}\,d\tilde{m}(t), \qquad L(s, t) = \int_X u(x, s)\,\overline{u(x, t)}\,dm(x)$$

[$R(x, y)$ being the kernel of U^*U and $L(s, t)$ that of UU^*]?

Lemma 2.1. An operator U_s has the same correlations as an operator U, namely

$$U_s U_s^* = UU^* \tag{2.10}$$

$$U_s^* U_s = U^*U \tag{2.11}$$

if and only if there exists a pair of unitary operators $(S, \tilde{S})$, S being defined on $H(X)$ and $\tilde{S}$ on $H(T)$, that is, there exists a spatiotemporal symmetry (see ref. 1 and Section 3 below) such that

$$U_s = US = \tilde{S}U \tag{2.12}$$

Remark 2.2. This result once again shows the importance of spatiotemporal symmetries. In particular, the existence of such a symmetry (2.12) immediately implies the identity of the correlations (2.10) and (2.11) (and vice versa). This explains why a spatiotemporal symmetry involving the S^1-symmetry defined by the rotation by $\exp(i\beta)$ is necessarily lost in the correlations but not in the operator U (see also Examples 5.3, 5.4, and 5.6 given in Section 5).

Proof. Equation (2.12) trivially implies (2.10) and (2.11).

Conversely, if (2.10) and (2.11) are true, then the polar decompositions of

$$V=\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \quad \text{and} \quad V_s=\begin{pmatrix} 0 & U_s^* \\ U_s & 0 \end{pmatrix}$$

into positive operators and partial isometries, $V=W\,|V|$ and $V_s=W_s\,|V_s|$, (see ref. 20 for the polar decomposition of U) are such that:

(i) $|V|=|V_s|$.

(ii) W and W_s have common domains and ranks.

If we write

$$W_s=\begin{pmatrix} 0 & G_s \\ \tilde{G}_s & 0 \end{pmatrix} \quad \text{and} \quad W=\begin{pmatrix} 0 & G \\ \tilde{G} & 0 \end{pmatrix}$$

we can then define $S=GG_s^{-1}$ on the common rank of G and G_s, and $S=\mathrm{Id}$ on the orthogonal complement. Similarly, we define $\tilde{S}=\tilde{G}_s\tilde{G}^{-1}$ on the common rank of $\tilde{G}$ and $\tilde{G}_s$ and $\tilde{S}=\mathrm{Id}$ on the orthogonal complement. Equation (2.12) can then be easily deduced.

Remark 2.3. The operators S and $\tilde{S}$ correspond to rotations inside the different eigenspaces. If the function $u(x, t)$ varies with a parameter λ, the possibility of such rotations inside degenerate eigenspaces of dimension higher than one may lead to a spatiotemporal bifurcation at a certain parameter value $\lambda=\lambda_c$. The term "bifurcation" is used here in analogy with Poincaré's terminology regarding temporal systems (see, e.g., ref. 39): we say that a space-time bifurcation occurs when there is a lack of smoothness in $u(x, t)$ as a function of a parameter, corresponding to a change in the qualitative (space-time) behavior of the solution. At $\lambda=\lambda_c$, we assume a degeneracy of eigenvalues which cannot be removed by small perturbations in the complex plane, while for $\lambda<\lambda_c$, the eigenvalues are nondegenerate. From the general perturbation theory of linear operators, we know that the (uniquely defined) eigenvectors smoothly vary with λ, for $\lambda<\lambda_c$. This may no longer be true at $\lambda=\lambda_c$, as the degenerate subspace is now spanned by equivalent eigenvectors and a rotation of the eigenvectors may occur. Since this perturbation theory is applicable to the operator V, rotations may occur independently among the topos and the chronos in the degenerate eigenspaces and are equivalent, in view of the present remark, to a change of the spatiotemporal symmetries of $u(x, t)$. This systematically leads to symmetry-involved bifurcations. [The occurrence of such a bifurcation is related to the existence of a cut—which is the projection in the complex plane of the intersection of the eigenvalue surfaces—in the complex plane of the extended parameter.[40, 41] If such a cut does not exist ("self-avoiding" case), the bifurcation does not take place.] As will become clear in Section 3.1, if, after and before $\lambda = \lambda_c$, the degeneracy of the eigenvalues of U is the same, the symmetries are different, but the two groups of symmetries $\Gamma_\lambda(U)$ are isomorphic; in contrast, if, after and before $\lambda = \lambda_c$, the degeneracy of the eigenvalues of U varies, the groups of symmetries are no longer isomorphic: in this case, we observe a symmetry-breaking (symmetry-increasing or -decreasing) bifurcation. An example is given by ref. 27 in the case of bifurcations of propagating waves through which newly formed traveling waves with different speeds appear in the solution: a particular case, given in Example 5.5 of Section 5, is furnished by the bifurcation of a traveling wave to the superposition of two traveling waves[27] for which the degeneracy stays the same before and after the bifurcation, so the groups of symmetries are isomorphic. The rotation of the eigenvectors, however, responsible for the appearance of a new traveling wave has changed the symmetries of the solution $u(x, t)$. The space-time translation symmetry characteristic of a pure traveling wave is then broken (see ref. 27 and Example 5.1 of Section 5).

It is important to note that the existence of a spatiotemporal symmetry is essential in the argument used in the proof of Lemma 2.1. Indeed, if one needs the coincidence of the spatial two-point correlations (2.11) and that of the temporal two-point correlations (2.10), the spatial or temporal symmetry alone does not suffice.

Remark 2.4. If we relax one of the two conditions, e.g., (2.11) [or (2.10)], and seek the functions having the same spatial two-point correlation $R(x, y)$ [or the same temporal two-point correlation $L(s, t)$], it then suffices to consider the temporal (or spatial) symmetry. More precisely, a necessary and sufficient condition for U_s and U to have the same spatial (resp. temporal) correlations is the existence of a temporal (resp. spatial) symmetry such that $U_s = \tilde{S}U$ (resp. $U_s = US$). Note that in this case, the temporal (resp. spatial) characteristic space $\chi(T)$ [resp. $\chi(X)$] of U and that of U_s do not necessarily coincide, in contrast to the situation of Lemma 2.1.

3. SYMMETRIES

3.1. The Space-Time Symmetry Group

The notion of spatiotemporal symmetry has been introduced in ref. 1 as a pair of operators $(S, \tilde{S})$, S defined on $H(X)$ and $\tilde{S}$ on $H(T)$, such that

$$US = \tilde{S}U \tag{3.1}$$

and whose conjugate pair $(\tilde{S}^*, S^*)$ also satisfies

$$\tilde{S}^* U = U S^* \tag{3.2}$$

Equivalently, we can write $VR = RV$, where

$$R = \begin{pmatrix} S & 0 \\ 0 & \tilde{S} \end{pmatrix}$$

Although it is possible to consider non unitary operators,[1, 18] for the sake of simplicity, we restrict ourselves here to unitary operators S and $\tilde{S}$.

Given the Hilbert spaces $H(X)$ and $H(T)$, in order to identify *all* the spatiotemporal symmetries of a given function, it is useful to introduce the notion of the spatiotemporal symmetry group of $u(x, t)$.

Definition 3.1. The spatiotemporal symmetry group of U is the set $\Gamma(U)$ of pairs of symmetries such that

$$\Gamma(U) = \{(S, \tilde{S}),\ S \text{ and } \tilde{S} \text{ unitary acting on } H(X) \text{ and } H(T), \text{ and } \tilde{S}^* U = US^*\} \tag{3.3}$$

$\Gamma(U)$ forms a group for the product:

$$(S_1, \tilde{S}_1) * (S_2, \tilde{S}_2) = (S_1 S_2, \tilde{S}_1 \tilde{S}_2)$$

since

$$\tilde{S}_1 \tilde{S}_2 U = \tilde{S}_1 U S_2 = U S_1 S_2$$

Theorem 3.2. Characterization of the spatiotemporal symmetry group. Given an operator U, the spatiotemporal symmetry group satisfied by U is

$$\Gamma(U) \cong L(U) \cap L_0(X, T) \cap \mathrm{Unit}(X, T) \tag{3.4}$$

where

$$L(U) = \{R \in L(H(X) \oplus H(T)),\ RV = VR\} \tag{3.5}$$

$$L_0(X, T) = \{R \in L(H(X) \oplus H(T)),$$
$$R(a1_X \oplus b1_T) = (a1_X \oplus b1_T)\, R,\ \forall a, b \in C\} \tag{3.6}$$

$$\mathrm{Unit}(X, T) = \{R \in L(H(X) \oplus H(T)),\ RR^* = R^*R = 1\} \tag{3.7}$$

Remark 3.3. $\Gamma(U)$ is the intersection of three different sets among which only one, $L(U)$, depends on U. The (two) other ones depend only on the Hilbert spaces $H(X)$ and $H(T)$. Obviously, the definition of $\Gamma(U)$

implicitly depends on the Hilbert structure used in the definition of the operator U. Note that the situation is similar to that met in the study of the solutions of partial differential equations (PDE), where the choice of an appropriate space of functions is essential.

Proof of Theorem 3.2. Suppose that $(S, \tilde{S}) \in \Gamma(U)$. By identifying $(S, \tilde{S})$ with the operator

$$R = \begin{pmatrix} S & 0 \\ 0 & \tilde{S} \end{pmatrix} \tag{3.8}$$

acting on $H(X) \oplus H(T)$, one can easily see that $R \in L(U) \cap L_0(X, T) \cap \mathrm{Unit}(X, T)$. Conversely, if $R \in L(U) \cap L_0(X, T) \cap \mathrm{Unit}(X, T)$, then

$$R = \begin{pmatrix} S & S_{12} \\ S_{21} & \tilde{S} \end{pmatrix}$$

is systematically of the form (3.8), namely $S_{12} = S_{21} = 0$, since it belongs to $L_0(X, T)$. Moreover, since $R \in L(U)$, we obtain

$$\tilde{S}U = US$$

$$SU^* = U^*\tilde{S}$$

Finally, S and $\tilde{S}$ are unitary because R itself is unitary. This shows that R can be identified with the pair $(S, \tilde{S}) \in \Gamma(U)$.

$\Gamma(U)$ is the largest group of spatiotemporal symmetries satisfied by U. $L_0(X, T)$ is the commutant of the von Neumann algebra of diagonable[3] operators in $H(X) \oplus H(T)$ and $L(U)$ that of the von Neumann algebra spanned by U. It follows that $L(U) \cap L_0(X, T)$ can be replaced by the commutant of the algebra spanned by V and diagonable operators, so that $\Gamma(U) = \{\text{unitary operators of } M(U)'\}$, where

$$M(U) = \{V, a1_X \oplus b1_T, \forall a, b\}''$$

where we have used the usual notations, M' for the commutant of M, and M'' for the commutant of M' (the von Neumann algebra spanned by M).

We now describe the group $\Gamma(U)$ in detail and give an algorithm to construct all the symmetries $(S, \tilde{S})$ satisfied by $u(x, t)$. This technique will be implemented on a concrete case in Example 5.1 of Section 5. For the sake of clarity, we suppose that the operator U has a discrete spectrum, which is the case, for instance, when U is compact. The generalization to

[3] Here, an operator is said to be diagonable if it is the direct sum of scalar operators.

the case of continuous spectra can be treated by following the strategy adopted in ref. 18 consisting in working in the space of generalized eigenvectors.

Notation 3.4. Let U be the operator whose kernel is the function $u(x, t)$ and whose spectrum is assumed to be discrete. We denote by A_k, $k = 0, 1,\ldots,$ the nonzero eigenvalues of U and d_k the dimension of the respective eigenspaces $(E_k, \tilde{E}_k)$. Obviously,

$$\chi(X) = \bigoplus_k E_k \subset H(X) \tag{3.9}$$

$$\chi(T) = \bigoplus_k \tilde{E}_k \subset H(T) \tag{3.10}$$

We now give the theorem of the group reconstruction.

Theorem 3.5. Reconstruction of the spatiotemporal symmetry group. We denote $U(d_k)$ the group of unitary operators of dimension d_k. Given the operator U, the spatiotemporal symmetry group $\Gamma(U)$ (introduced in Definition 3.1) is isomorphic to the direct product of the groups $U(d_k)$, namely

$$\Gamma(U) \approx \prod_k U(d_k) \tag{3.11}$$

where Notation 3.4 has been used. Therefore, each spatiotemporal symmetry $(S, \tilde{S})$ in $\Gamma(U)$ is fully specified by the selection of an orthonormal basis in each eigenspace E_k of U (which are then the eigenvectors of S) and the choice of a family of real numbers in $[0, 1[$, $\theta_k(1), \theta_k(2),\ldots, \theta_k(d_k)$ $\{\lambda_k(s) = \exp[2i\pi\theta_k(s)]$ being the eigenvalues of $S\}$, for each k.

Proof. The elements $(S, \tilde{S})$ of $\Gamma(U)$ are such that S restricted to E_k (resp. $\tilde{S}$ restricted to $\tilde{E}_k$) is a unitary operator and, conversely, each unitary operator on $\chi(X)$ which commutes with the projections onto each eigenspace E_k gives rise to a spatiotemporal symmetry in $\Gamma(U)$. This last property is due to the fact that the operator U realizes an isomorphism between $\chi(X)$ and $\chi(T)$,[20] and therefore $\tilde{S}$ is systematically fixed by S. Since the relation $US = \tilde{S}U$ makes the actions of S (resp. $\tilde{S}$) on the different eigenspaces E_k (resp. $\tilde{E}_k$) independent, the first assertion of the theorem then follows. The second part is simply a constructive counterpart of the first one.

Remark 3.6. Given the Hilbert spaces $H(X)$ and $H(T)$, the previous theorem gives a recipe for the construction of *all* the spatiotemporal symmetries of U. That is, if we express S (resp. $\tilde{S}$) in the basis of the topos (resp. chronos) of the operator U, we obtain a block operator defined by any unitary transformation acting on each eigenspace of U in $\chi(X)$ [resp. $\chi(T)$]. Equivalently, S (resp. $\tilde{S}$) is defined by the choice of an orthonormal basis of eigenvectors in each subspace E_k (resp. $\tilde{E}_k$) and, for each of these subspaces, by the choice of the corresponding d_k eigenvalues $\lambda_k(s) = \exp[2i\pi\theta_k(s)]$, $s = 1,..., d_k$. Of course, as is well known, the group $U(d_k)$ has less parameters than those indicated here since, in order to specify $S \in U(d_k)$, it suffices to fix one oriented basis, namely a Hermitian matrix of trace zero, and one phase θ. Instead, a single symmetry S reconstructed with the previous recipe may correspond to various choices of our parametrizations, but this freedom leads to a "natural" choice in practice (see Example 5.1 in Section 5).

Among all the possible spatiotemporal symmetries $(S, \tilde{S})$, it is interesting to identify those with a spatial (resp. temporal) component S (resp. $\tilde{S}$) which is induced by an automorphism of the physical space X (resp. T), namely the so-called point symmetries.

Definition 3.7. S (resp. $\tilde{S}$) is called a point symmetry if it is induced by a measurable automorphism of X (resp. T), namely if there exists a measurable, invertible transformation $f: X \to X$ (resp. $g: T \to T$) such that

$$(S\phi)(x) = \frac{1}{a}\,\phi(f^{-1}(x)) \tag{3.12}$$

where a is a nonzero constant (resp.

$$(\tilde{S}\psi)(t) = \frac{1}{b}\,\psi(g^{-1}(t)) \tag{3.13}$$

where b is a nonzero constant).

We denote $\Gamma_0(U)$ the subgroup of the point space-time symmetries of $\Gamma(U)$. Note that allowing the constants a and b in (3.12) and (3.13) to be different from one permits the consideration of automorphisms which do not preserve the measures $dm(x)$ and $d\tilde{m}(t)$. We now use the next theorem to identify $\Gamma_0(U)$, which is the transcription of the Multiplication Theorem in our context, and therefore we refer to ref. 42 for its proof.

Theorem 3.8. S is a point symmetry if and only if S and S^{-1} transform any bounded function into a bounded function and

$$S(\phi_1 \cdot \phi_2) = S\phi_1 \cdot S\phi_2 \tag{3.14}$$

whenever ϕ_1 and ϕ_2 are bounded functions, where the pointwise product is defined by

$$\forall x, \quad (\phi_1 \cdot \phi_2)(x) = \phi_1(x)\, \phi_2(x) \tag{3.15}$$

The same result is true if we replace S by $\tilde{S}$ and $\phi_i(x)$ by $\psi_i(t)$, $i = 1, 2$.

Remark 3.9. In the case where $H(X) = L^2(R)$ and $H(T) = L^2(R)$, we proved in ref. 1 that the only point symmetries implemented by differentiable transformations are dilationtranslations where the transformations f and g in (3.12) and (3.13) are

$$f(x) = \pm a^2 x - x_0 \tag{3.16}$$

$$g(t) = \pm b^2 t - t_0. \tag{3.17}$$

where a and b are the same constants as in (3.12) and (3.13). Therefore, if $a = b = 1$ in (3.16) and (3.17), the only differentiable point symmetries belong to the group of translations and reflections. An example of a bifurcation through which the solution is no longer invariant under Γ_0 is given in Example 5.5 in Section 5.

3.2. Symmetries of the Orbits in $\chi(X)$ and $\chi(T)$

In this subsection we show when the existence of a spatiotemporal symmetry $(S, \tilde{S})$ implies that the orbit in $\chi(X)$ is itself invariant under S [or the orbit in $\chi(T)$ is itself invariant under $\tilde{S}$] and conversely. We recall that the (time) orbit of the dynamical system is defined as the set $\{\xi_t, t \in T\}$ in $\chi(X)$ defined as

$$\xi_t(x) = u(x, t), \qquad \forall x \in X \tag{3.18}$$

and the space orbit as the set $\{\eta_x, x \in X\}$ in $\chi(T)$ defined as

$$\eta_x(t) = u(x, t), \qquad \forall t \in T \tag{3.19}$$

Proposition 3.10. The pair of operators $(S, \tilde{S})$ is a spatiotemporal symmetry of U where $\tilde{S}$ (resp. S) is a point symmetry [namely it is implemented on the time (resp. space) variable] if and only if the time orbit ξ_t in $\chi(X)$ [resp. the space orbit η_x in $\chi(T)$] is invariant under S (resp. $\tilde{S}$), i.e.,

$$S\{\xi_t, t \in T\} = \{\xi_t, t \in T\} \tag{3.20}$$

(resp.

$$\tilde{S}\{\eta_x, x \in X\} = \{\eta_x, x \in X\}\,) \tag{3.21}$$

Proof. Let us define $\tilde{S}_g$ by

$$\forall \psi \in H(T), \quad \forall t, \qquad (\tilde{S}_g \psi)(t) = \psi(g^{-1}(t))$$

for an invertible, measurable function g. By computing the following scalar product of the image of ξ_t by S with any function in $H(X)$, we obtain

$$\begin{aligned} \forall t, \quad \forall \varphi \in H(X), \qquad (S\xi_t, \bar{\varphi}) &= (\xi_t, S^{-1}\bar{\varphi}) \\ &= \int u(x, t)(S^{-1}\varphi)(x)\, dm(x) \\ &= (US^{-1}\varphi)(t) \\ &= (\tilde{S}_g^{-1} U\varphi)(t) \\ &= (U\varphi)(g(t)) \\ &= (\xi_{g(t)}, \bar{\varphi}) \end{aligned}$$

The invariance of the orbit (3.20) immediately follows.

Conversely, if the orbit is invariant under S,(3.20), then S preserves $\chi(X)$, since the latter is the smallest linear space containing $\{\xi_t(x), \forall t\}$,[20] i.e., $S\chi(X) = \chi(X)$. This is due to the fact that

$$\forall \varphi \in \chi(X), \qquad \varphi = \lim_{N \to \infty} \sum_{k=1}^{N} \alpha_k \xi_{t_k}$$

and therefore

$$S\varphi_N = \sum_{k=1}^{N} \alpha_k S\xi_{t_k} \in \chi(X)$$

Then

$$\forall t, \quad \exists g(t), \quad \forall \varphi \in H(X), \qquad (S\xi_t, \bar{\varphi}) = (\xi_{g(t)}, \bar{\varphi})$$

Using the same computation as above and since (3.20) implies that g is an invertible, measurable function, we obtain

$$US^{-1} = \tilde{S}^{-1} U$$

The proof is similar for the symmetry of the space orbit η_x.

In the proof of Proposition 3.10, one can see the importance of choosing the phase space $\chi(X)$ (the smallest linear space containing the dynamics), rather than a larger one. Examples where the orbits in $\chi(X)$ and $\chi(T)$ are invariant by a symmetry are given in Section 5, Examples 5.1 5.2, and 5.4.

Remark 3.11. If, instead of having a spatiotemporal symmetry, one has a spatiotemporal quasi-symmetry,[1] defined by a pair of unitary operators such that

$$US = \gamma \tilde{S} U$$

$$US^* = \gamma^{-1} \tilde{S}^* U$$

where γ is a real number different from 1, the proof of Lemma 3.13 still works if the coefficient γ is introduced by using the new operator commutation relation. It follows that

$$\forall t, \quad \forall \varphi \in H(X), \qquad (S\xi_t, \varphi) = \gamma^{-1}(\xi_{g(t)}, \varphi)$$

which means that the symmetric image of the orbit in $\chi(X)$ is the orbit dilated (or compressed) by the factor γ^{-1}, and similarly for the orbit in $\chi(T)$. Such is the situation when S and $\tilde{S}$ are dilation symmetries, as in fully developed turbulence.[18, 24] Note that the structure of the quasi-symmetry group is very different from that of a symmetry group, described in Theorem 3.2, since now each spatiotemporal symmetry acts from one pair of eigenspaces $(E_k, \tilde{E}_k)$ to another pair $(E_{k'}, \tilde{E}_{k'})$ with $k \neq k'$. This precisely led to the derivation of spectral laws and a self-similar spatiotemporal structure in turbulence.[18,24]

3.3. Symmetries of the Time and Space Averages

In this subsection we investigate under which conditions a spatiotemporal symmetry implies an invariance of the time average (resp. the space average). Here, we consider the case where $\tilde{m}(T) < \infty$ [resp. $m(X) < \infty$] since then it is possible to treat the problem inside $H(T)$ [resp. $H(X)$]. It is clear from the proof of Lemma 3.13 (see below) that this condition can be relaxed if one replaces the time average defined in (3.22) [resp. the space average defined in (3.29)] by a time (resp. space) limit procedure, provided we require that $u(x, t)$ satisfies a sufficient uniform continuity condition which will permit the inversion of the limits in this proof. However, we prefer to restrict ourselves in this subsection to the former (simpler) case, which still permits the treatment of most applications and we refer the reader to refs. 18 and 43 for the generalization to infinite measures.

For any spatiotemporal function $u(x, t) \in L^1(T, d\tilde{m})$, for m-a.e. $x \in X$, we define its time average as

$$M_t(x) = \int_T u(x, t)\, d\tilde{m}(t) \tag{3.22}$$

Definition 3.12. We define the projector

$$\tilde{P}: H(T) \to H(T)$$

such that

$$(\tilde{P}\psi)(t) = \int_T \psi(t)\, d\tilde{m}(t) \tag{3.23}$$

for any $\psi \in L^1(T, d\tilde{m}) \cap H(T)$, and similarly the projector

$$P: H(X) \to H(X)$$

such that

$$(P\varphi)(x) = \int_X \varphi(x)\, dm(x) \tag{3.24}$$

for any $\phi \in L^1(X, dm) \cap H(X)$.

Notice that $\tilde{P}\psi$ defined on $H(T)$ [resp. $P\phi$ defined on $H(X)$], is a constant function that is independent of t (resp. x). Moreover, under the given finite-weight condition assumed for the measures, the operators P and $\tilde{P}$ are two orthogonal projectors, namely $P = P^2 = P^*$ and $\tilde{P} = \tilde{P}^2 = \tilde{P}^*$.

Lemma 3.13. If $u(x, t) \in L^1(T, d\tilde{m})$, for m-a.e. $x \in X$, satisfies a spatio-temporal symmetry $(S, \tilde{S})$, then $\tilde{S}$ fulfills the condition

$$\int_T (\tilde{S}\eta_x)(t)\, d\tilde{m}(t) = \int_T \eta_x(t)\, d\tilde{m}(t) \tag{3.25}$$

if and only if the time average of $u(x, t)$, $M_t(x)$, is invariant under S, i.e.,

$$SM_t = M_t \tag{3.26}$$

Proof. Since $\chi(T)$ is the closed linear span of $\{\eta_x, x \in X\}$,[18] we see that (3.25) is equivalent to $\tilde{S}\tilde{P} = \tilde{P}$ on $\chi(X)$. Now, we can write, on one hand,

$$(M_t, \bar{\varphi}) = \int_T (U\varphi)(t)\, d\tilde{m}(t) \tag{3.27}$$

and, on the other hand,

$$(SM_t, \bar{\varphi}) = \int_T (US^{-1}\varphi)(t)\, d\tilde{m}(t)$$

$$= \int_T (\tilde{S}^{-1}U\varphi)(t)\, d\tilde{m}(t) \tag{3.28}$$

Since the rank of U is $\chi(T)$, the equality of (3.27) and (3.28) is equivalent to

$$\forall \psi \in \chi(T), \qquad \int_T \psi(t)\, d\tilde{m}(t) = \int_T (\tilde{S}^{-1}\psi)(t)\, d\tilde{m}(t)$$

which ends the proof.

Remark 3.14. On one hand, the condition concerning the orbit in phase space used in the previous lemma is weaker than that used in Proposition 3.10 (since the former is only a condition regarding the integral of the orbit). On the other hand, the fact that the spatiotemporal symmetry is an *a priori* hypothesis in the previous lemma is a strong condition compared to Preposition 3.10, where it follows from the hypothesis. Note that the condition (3.25) is systematically satisfied if $\tilde{S}$ is a point symmetry, namely if it acts on $\psi(t)$ by an action on the variable t, since $\tilde{P}\psi$ is a constant function.

Remark 3.15. Lemma 3.13 is still valid if space and time are interchanged (the proof is similar). We can then write the counterpart of Lemma 3.13 as follows. Suppose that $u(x, t)$ satisfies a space-time symmetry $(S, \tilde{S})$. Then the operator S satisfies the condition

$$\int_X (S\xi_t)(x)\, dm(x) = \int_X \xi_t(x)\, dm(x)$$

or, equivalently,

$$SP = P \qquad \text{on} \quad \chi(X) \tag{3.29}$$

(that is, S leaves the constant functions invariant) if and only if the space average of the function $u(x, t)$

$$M_x(t) = \int_X u(x, t)\, dm(x) \tag{3.30}$$

where $u(x, t) \in L^1(X, dm)$ for $\tilde{m}$-a.e. $t \in T$, is invariant under $\tilde{S}$, i.e.,

$$\tilde{S}M_x = M_x \tag{3.31}$$

As in Remark 3.14 for the condition regarding $\tilde{S}$, the relation (3.29) is systematically satisfied if S is a point symmetry.

3.4. Symmetries of the Correlation Operators

Regarding the symmetry issue, the major difference between the operator U^*U (or similarly UU^*) and the operator U lies again in the fact that the former acts inside a Hilbert space while the latter does not. Nevertheless, the classical notion of a symmetry acting on a single Hilbert space can be recovered in two ways: (a) by identifying S (or $\tilde{S}$) with the identity operator, namely $US = U$ (or $\tilde{S}U = U$), in which case the signal is invariant under the symmetry at every time t, that is, instantaneously (or at every location x, that is, uniformly); (b) in a statistical sense. For the latter, it suffices to note that the commutation of U with a spatiotemporal symmetry $(S, \tilde{S})$, expressed in (3.1) and (3.2) implies the commutation of U^*U with the spatial symmetry S and the commutation of UU^* with the temporal symmetry $\tilde{S}$, i.e.,

$$U^*US = SU^*U \tag{3.32}$$

$$UU^*\tilde{S} = \tilde{S}UU^* \tag{3.33}$$

[In the case of a spatiotemporal quasi-symmetry as in Remark 3.11, we simply have to multiply the right-hand sides of (3.32) and (3.33) by the factor γ^2.[1, 18]

Obviously, Eqs. (3.32) and (3.33) do not imply (3.1) and (3.2). More precisely, the number of pairs $(S, \tilde{S})$ satisfying (3.32), (3.33) is much larger than the number of pairs satisfying (3.1), (3.2). This is due to the fact that S in $\chi(X)$ and $\tilde{S}$ in $\chi(T)$ are coupled through the isomorphism U in (3.1), (3.2) but not in (3.32) and (3.33). Therefore, a symmetry of (3.1), (3.2) is stronger than a symmetry (3.32), (3.33). Hereafter, we say that S (resp. $\tilde{S}$) is a statistical spatial (resp. temporal) symmetry if U^*U (resp. UU^*) commutes with S, namely (3.32) [resp. (3.33)] is fulfilled. Nevertheless, if there exists an operator S (resp. $\tilde{S}$) such that (3.32) [resp. (3.33)] is satisfied, then there exists an operator $\tilde{S}$ (resp. S) such that a spatiotemporal symmetry (3.1), (3.2) holds.[18] An example of two functions satisfying the same statistical symmetries but different spatiotemporal symmetries is given in Section 5 (Example 5.4).

4. IDENTIFICATION OF FUNCTIONS WITH GIVEN SPATIOTEMPORAL SYMMETRIES

In this section, we give a solution to the inverse problem to that resolved in Theorem 3.2, namely we construct *all* the space-time functions which admit a given group of spatiotemporal symmetries. Obviously, if we wish that this problem possesses other solutions than the trivial one, i.e., $u(x, t) = 0$, the group $\Gamma = \{(S, \tilde{S})\}$ must act spatially and temporally in a compatible manner. More precisely, $\Gamma = \{(S, \tilde{S})\}$ must be such that it is possible to find *nontrivial* subspaces $\chi_\Gamma(X) \subset H(X)$ invariant by all the spatial symmetries S, and *nontrivial* subspaces $\chi_\Gamma(X) \subset H(X)$ invariant by all the temporal symmetries $\tilde{S}$ and such that the representation of Γ induced by the operators S on $\chi_\Gamma(X)$ and that induced by the operators $\tilde{S}$ on $\chi_\Gamma(T)$ are equivalent. By a simple application of Zorn's Lemma, there exist two such subspaces $\chi_\Gamma(X)$ and $\chi_\Gamma(T)$ which are maximal.

As we will see below, the reconstructed functions $u(x, t)$ will be trivial on the orthogonal complements of $\chi_\Gamma(X)$ and $\chi_\Gamma(T)$. It turns out that a spatiotemporal symmetry group acts as equivalent representations on the space and time domains. This property, already mentioned in ref. 1, emphasizes the fact that there is only "one" action of the group Γ and this action is simultaneous in space and time.

Definition 4.1. Given a group of spatiotemporal unitary symmetry operators $\Gamma = \{(S, \tilde{S})\}$, we define $U(\Gamma)$ as the set of operators U having as a space-time group of symmetries, namely

$$U(\Gamma) = \{U,\ US = \tilde{S}U \text{ and } \tilde{S}^* U = US^*,\ \forall (S, \tilde{S}) \in \Gamma\} \tag{4.1}$$

The algebraic characterization of $U(\Gamma)$ is rather simple since it is the S-right as well as the $\tilde{S}$-left module generated by any $U \in U(\Gamma)$, in particular by the unitary intertwining operator of the space and time representations of Γ described above (see ref. 44 for details). We now describe an effective algorithm for constructing all the functions $u(x, t)$, such that the corresponding operator U belongs to $U(\Gamma)$.

Proposition 4.2. Let $\Gamma = \{(S, \tilde{S})\}$ be a group of unitary space-time symmetries. Then all the functions $u(x, t)$ for which the corresponding operator U satisfies the spatiotemporal symmetry group, i.e., $\Gamma(U) = \Gamma$, is given by the following algorithm:

(1) Decompose $H(X)$ and $H(T)$ as the finest possible partition such that

$$H(X) = \bigoplus_l E_l \oplus \chi_\Gamma(X)^\perp \tag{4.2}$$

$$H(T) = \bigoplus_l \tilde{E}_l \oplus \chi_\Gamma(X)^\perp \tag{4.3}$$

with

$$\dim E_l = \dim \tilde{E}_l = d_l, \qquad \forall l \tag{4.4}$$

and

$$SE_l \subset E_l, \qquad \forall l, \quad \forall S \tag{4.5}$$

$$\tilde{S}\tilde{E}_l \subset \tilde{E}_l, \qquad \forall l, \quad \forall \tilde{S} \tag{4.6}$$

for all $(S, \tilde{S}) \in \Gamma$

(2) Choose a (any) partition of the indices l, and for any element of the partition $K = \{l_{i_1},..., l_{i_n},...\}$ denote by E_K and $\tilde{E}_K$ the corresponding spaces:

$$E_K = \bigoplus_{l \in K} E_l \tag{4.7}$$

$$\tilde{E}_K = \bigoplus_{l \in K} \tilde{E}_l \tag{4.8}$$

(3) Choose a (any) family of nonnegative numbers indexed by K, $\{A_K \geqslant 0\}$.

(4) For each K, choose a (any) linear equivariant isomorphism T_K from E_K onto $\tilde{E}_K$, namely a unitary operator such that

$$\tilde{S}T_K = T_K S$$

(5) Define U by

$$U\varphi = A_K(T_K \varphi)$$

for any $\varphi \in E_K$, or equivalently

$$u(x, t) = \sum_{\substack{K \\ n(K) \in K}} A_K \varphi^{(K)}_{n(K)}(x)\, \psi^{(K)}_{n(K)}(t)$$

where

$$\psi^{(K)}_{n(K)} = T_K \varphi^{(K)}_{n(K)}$$

Proof. To prove that $US = \tilde{S}U$, by using point (5), we simply write

$$U = \bigoplus_K A_K T_K$$

We now add some additional remarks. First, the partition described in point (1) of the algorithm always exists, but in some cases it may be trivial, namely

$$E_l = H(X), \qquad \tilde{E}_l = H(T)$$

In this case, $U(\Gamma)$ is also trivial. At the other extreme, the partition is the finest possible one, namely all the subspaces E_l and all the subspaces $\tilde{E}_l$, are one dimensional. An interesting example of the latter case is a locally compact Abelian group where the symmetries $\{(S, \tilde{S})\}$ define irreducible (equivalent) representations on each E_l and $\tilde{E}_l$. It is important to point out that such a partition depends only on $\Gamma = \{(S, \tilde{S})\}$ and, according to Theorem 3.2, it fixes the finest block structure of U. It is also clear that, for a given U, we have

$$\chi_U(X) = \bigoplus_{\substack{K \\ A_K \neq 0}} E_K \subset \chi_\Gamma(X)$$

and

$$\chi_U(T) = \bigoplus_{\substack{K \\ A_K \neq 0}} \tilde{E}_K \subset \chi_\Gamma(T)$$

Therefore, one can see that the only nontrivial step in the construction of the functions $u(x, t)$ is the investigation of the structure of the irreducible representations which decompose Γ into $\chi_\Gamma(X)$ and $\chi_\Gamma(T)$ and that of the intertwining operators. Once this structure is known, the functions $u(x, t)$ are completely determined by the "energies" A_K of the corresponding irreducible representations of Γ on $\chi_\Gamma(X)$ and $\chi_\Gamma(T)$.

5. EXAMPLES

We now give examples which illustrate Sections 2–4.

5.1. Spatiotemporal Symmetries

Example 5.1. An example of a function satisfying a space-time symmetry is a traveling wave defined by

$$u(x - x_0, t) = u(x, t + t_0)$$

for all $x, x_0 \in X$, $t, t_0 \in T$ such that $x_0 + ct_0 = 0$, c being the propagation speed of the wave. We know that this property[1] is equivalent to the operator intertwining relation:

$$US_{x_0} = \tilde{S}_{t_0} U \qquad \text{and} \qquad U\tilde{S}^*_{t_0} = S^*_{x_0} U \tag{5.1}$$

where S_{x_0} is the regular representation of R on $H(X) = L^2(X)$ defined by

$$(S_{x_0}\varphi)(x) = \varphi(x - x_0) \tag{5.2}$$

and $\tilde{S}_{t_0}$ the regular representation of R on $H(T) = L^2(T)$ defined by

$$(\tilde{S}_{t_0}\psi)(t) = \psi(t - t_0) \tag{5.3}$$

with the relation $x_0 + ct_0 = 0$. This obviously implies that the spatial two-point correlation function is invariant under S_{x_0} (cf. Section 3.4),

$$\forall x_0 \in R, \quad S_{x_0} U^* U = U^* U S_{x_0} \tag{5.4}$$

and the temporal two-point correlation is invariant under $\tilde{S}_{t_0}$:

$$\forall t_0 \in R, \quad \tilde{S}_{t_0} U U^* = U U^* \tilde{S}_{t_0} \tag{5.5}$$

In addition, since $\tilde{S}_{t_0}$ leaves any constant function invariant, the temporal mean of $u(x, t)$ is invariant under S_{x_0} and, similarly, since S_{x_0} leaves any constant function invariant, the spatial mean is invariant under $\tilde{S}_{t_0}$ (cf. Lemma 3.13 and Remark 3.14).

We now compute all the spatiotemporal symmetries of U in this case, in order to show how Theorem 3.2 works. Recall that, associated with a traveling wave in a finite domain is a periodic function $g(z)$ such that

$$u(x, t) = g(x - ct) \tag{5.6}$$

In addition, we now suppose that g is a real function. The biorthogonal decomposition of a real traveling wave is the space-time Fourier decomposition with the dispersion relation induced by g,[1] namely

$$\begin{aligned} u(x, t) = \sum_k c_k [&\sin(2\pi kx)\cos(2\pi kct + \phi_k) \\ &- \cos(2\pi kx)\sin(2\pi kct + \phi_k)] \end{aligned} \tag{5.7}$$

where the eigenvalues c_k and the phases ϕ_k are related to the Fourier coefficients g_k of g by

$$c_k = 2\,|g_k| \tag{5.8}$$

$$\tan(\phi_k) = -\frac{\mathrm{Im}(g_k)}{\mathrm{Re}(g_k)} \tag{5.9}$$

Therefore, in the generic case, the spaces E_k and $\tilde{E}_k$ of Theorem 3.2 have dimension $d_k = 2$. Following Theorem 3.2, any pair $(S, \tilde{S}) \in \Gamma(U)$ is then fully defined by a choice of two parameters $\theta_k(1)$ and $\theta_k(2)$ in $[0,1[$ and a selection of an orthonormal basis $\{\varphi_1^k, \varphi_2^k\}$ in each eigenspace E_k, which we parametrize as follows (see Remark 3.6 for a comment on such a parametrization):

$$\phi_1^{(k)} = a_k \exp(2i\pi kx) - b_k \exp(-2i\pi kx) \tag{5.10}$$

$$\varphi_2^{(k)} = \overline{b_k} \exp(2i\pi kx) + \overline{a_k} \exp(-2i\pi kx) \tag{5.11}$$

with

$$a_k = \cos(\alpha) \exp[2i\pi(\varphi - \phi)]$$

$$b_k = \sin(\alpha) \exp(-2i\pi\varphi)$$

We then define the operator S in the following way:

$$S\varphi_1^{(k)} = \exp[2i\pi\theta_k(1)]\ \varphi_1^{(k)} \tag{5.12}$$

$$S\varphi_2^{(k)} = \exp[2i\pi\theta_k(2)]\ \varphi_2^{(k)} \tag{5.13}$$

for all k. The corresponding temporal symmetry, obtained from the dispersion relation, gives rise to the orthonormal basis $\{\psi_1^{(k)}, \psi_2^{(k)}\}$ of $\tilde{E}_k$, i.e.,

$$\psi_1^{(k)}(t) = a_k \exp(2i\pi ckt + \phi_k) - b_k \exp(-2i\pi ckt + \phi_k) \tag{5.14}$$

$$\psi_2^{(k)}(t) = \overline{b_k} \exp(2i\pi ckt + \phi_k) + \overline{a_k} \exp(-2i\pi ckt + \phi_k) \tag{5.15}$$

The operator $\tilde{S}$ is then defined by

$$\tilde{S}\psi_1^{(k)} = \exp[2i\pi\theta_k(1)]\ \psi_1^{(k)} \tag{5.16}$$

$$\tilde{S}\psi_2^{(k)} = \exp[2i\pi\theta_k(1)]\ \psi_2^{(k)} \tag{5.17}$$

Now, by introducing the expression (5.10) of $\phi_1^{(k)}$ in (5.12) and that (5.11) of $\phi_2^{(k)}$ in (5.13), we immediately identify S_{x_0} if all coefficients b_k are zero and $\theta_k^{(1)} = \theta_k^{(2)} = kx_0 \pmod 1$. The corresponding temporal symmetry $\tilde{S}$ is of course $\tilde{S}_{t_0}$ such that $x_0 + ct_0 = 0$. $\Gamma(U)$ contains also the spatial reflection symmetry R_x and the temporal reflection symmetry $\tilde{R}_t$, but this pair does not belong to the same space-time symmetry, namely the time (resp. space) companion of R_x (resp. $\tilde{R}_t$) is not $\tilde{R}_t$ (resp. R_x). We now consider this problem in detail.

First, it is clear that, from (5.10), (5.11) and (5.12), (5.13), we recover the spatial reflection symmetry R_x, defined as

$$(R_x\varphi)(x) = \varphi(-x) \tag{5.18}$$

by taking

$$\forall k, \qquad a_k = b_k = \frac{1}{\sqrt{2}} \quad \text{and} \quad \theta_k(1) = \frac{1}{2}, \quad \theta_k(2) = 0 \tag{5.19}$$

This symmetry is, of course, satisfied by the spatial two-point correlation as well (see Section 3.4). Conversely, the invariance of the spatial correlation through the reflection symmetry (or any other symmetry) (as noticed, for instance, in ref. 45) should not be considered as a statistical artifact: such symmetry corresponds to the existence, for the space-time function itself, of a spatiotemporal symmetry whose temporal component can be computed, as we now show for the particular case of a traveling wave.

We see that the time symmetry corresponding to (5.18), (5.19), obtained from (5.14)–(5.17), using the same values for the parameters a_k, b_k, $\theta_k(1)$, and $\theta_k(2)$, has the following property. It leaves invariant the chronos corresponding to even topos and acts by a rotation of π on the chronos corresponding to odd topos. (This operator corresponds to the translation by a half period.) The same is, of course, true if we start with the temporal reflection symmetry $\tilde{R}_t$. On the one hand, following Section 3.1, we see that even if the space and time correlation functions are invariant by reflection (Section 3.4), these two symmetries do not form a companion pair $(S, \tilde{S})$ of the same space-time symmetry. On the other hand, the fact that the spatial two-point correlation is invariant by reflection is not an artifact, it does correspond to the existence of a space-time symmetry for $u(x, t)$ itself.

Example 5.2. Another example is furnished by the Karman street flow behind a cylinder (described in the introduction) subject to a spatiotemporal Z_2-symmetry group. To illustrate our point, we consider the streamwise velocity component as a function of the normal variable (x_2) which satisfies the spatiotemporal symmetry

$$u_1(x_2, t) = u_1(x_2, t - T/2) \tag{5.20}$$

(where T is the temporal period of the street), which is equivalent to the operator intertwining relations

$$US = \tilde{S}U \qquad \text{and} \qquad U^*\tilde{S} = SU^* \tag{5.21}$$

where S and $\tilde{S}$ are defined by

$$(S\varphi)(x_2) = \varphi(-x_2) \qquad \text{(reflection symmetry)} \tag{5.22}$$

$$(\tilde{S}\psi)(t) = \psi(t - T/2) \tag{5.23}$$

Again, the existence of the spatiotemporal symmetry, as well as the fact that $\tilde{S}$ acts on the time variable (and therefore leaves the constant functions invariant), imply that both the spatial two-point correlation and the time average of $u_1(x_2, t)$ are invariant under the reflection (Z_2-) symmetry group (5.22), as is well known in fluid mechanics. For similar reasons (see Section 3), the temporal two-point correlation and the space average of $u_1(x_2, t)$ are invariant under the (Z_2-) symmetry group (5.23). Moreover, the conditions of Proposition 3.10 are satisfied, so that the orbit in $\chi(X)$ has the spatial symmetry S and the orbit in $\chi(T)$ has the temporal symmetry $\tilde{S}$.

5.2. Spatiotemporal, Statistical, and Average Symmetries

We now give examples which illustrate how the various symmetries differ.

Example 5.3. Our first example in this category consists of the function $u(x, t)$ defined on $X = \{x_1\}$, $T = \{t_1\}$ such that

$$u(x_1, t_1) = \exp(i\alpha) \tag{5.24}$$

so that the operators U^*U, UU^* and V are

$$U^*U = \mathbf{1}_X, \qquad UU^* = \mathbf{1}_T \tag{5.25}$$

$$V = \begin{pmatrix} 0 & \exp(-i\alpha) \\ \exp(i\alpha) & 0 \end{pmatrix} \tag{5.26}$$

On one hand, we consider the eigenvalues and eigenfunctions of V which satisfy the equations

$$V\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = A\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

namely

$$\begin{cases} \exp(-i\alpha)\, \psi = A\varphi \\ \exp(i\alpha)\, \varphi = A\psi \end{cases} \tag{5.27}$$

from which we immediately deduce that the eigenvalues are 1 and -1. For $A = 1$, we can choose $\psi_1 = 1$, which implies that $\phi_1 = \exp(-i\alpha)$; for $A = -1$, we can choose $\psi_2 = 1$, which implies that $\phi_2 = -\exp(-i\alpha)$. We conclude that the phase is located in the "dispersion" relation

$$\phi_1 \leftrightarrow \psi_1, \quad \phi_2 \leftrightarrow \psi_2$$

On the other hand, we consider the correlation operators U^*U and UU^* for which the eigenvalue/eigenvector problem becomes:

$$U^*U\phi = A^2\phi \tag{5.28}$$

namely,

$$\mathbf{1}_X\phi = A^2\phi$$

so that $\phi = \exp(i\theta)$ with any θ and

$$UU^*\psi = A^2\psi \tag{5.29}$$

namely,

$$\mathbf{1}_T\psi = A^2\psi$$

so that $\psi = \exp(i\theta')$ with any θ'. It follows that the isomorphism between φ and ψ is lost.

Example 5.4. The following examples illustrate the fact that the statistical symmetries of two functions can be identical, but the deterministic spatiotemporal symmetries different.

(a) For this, we first consider a traveling wave $u_1(x, t)$ of velocity c, whose spatiotemporal symmetry, in terms of the corresponding operator U_1, is

$$\forall t_0, \quad U_1 S_{-ct_0} = \tilde{S}_{t_0} U_1 \tag{5.30}$$

and a traveling wave $u_2(x, t)$ of velocity $2c$, whose spatiotemporal symmetry, in terms of the corresponding operator U_2, is

$$\forall t_0, \quad U_2 S_{-2ct_0} = \tilde{S}_{t_0} U_2 \tag{5.31}$$

For both waves ($i = 1, 2$), the statistical symmetries are

$$\begin{aligned} &\forall x_0, \quad S_{x_0}^{-1} U_i^* U_i S_{x_0} = U_i^* U_i \\ &\forall t_0, \quad S_{t_0}^{-1} U_i U_i^* S_{t_0} = U_i U_i^* \end{aligned} \tag{5.32}$$

In this case, the statistical symmetries are identical but the correlations themselves are different. Moreover, the spatial means are invariant under spatial translation and the temporal means are invariant under temporal translations.

(b) An example dealing with spatiotemporal symmetries which are not pointwise is furnished by functions which are spatially and temporally modulated with respect to a reference dynamics $u_o(x, t)$, namely

$$u_{12}(x, t) = g_1(x)\, g_2(t)\, u_0(x, t) \tag{5.33}$$

where g_1 and g_2 are the spatial and temporal modulations. We suppose that the latter are real functions, different from constants. Assuming that $u_0(x, t)$ satisfies some space-time symmetry, one may wonder what remains from such symmetry in the modulated dynamics. We have shown that under certain conditions the space-time symmetry is deformed but not broken and the space-time symmetry of the modulated function can be expressed in terms of the symmetry present without modulation.[28] This was achieved by expressing the dynamic operator V_{12} associated with u_{12} in terms of the operator V_0 associated with u_0 by introducing the modulation matrix operator

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \tag{5.34}$$

such that the operators G_1 and G_2 are defined as

$$\begin{aligned} G_1 &: H(X) \to H(X) \\ [G_1(\phi)](x) &= g_1(x)\, \phi(x) \end{aligned} \tag{5.35}$$

$$\begin{aligned} G_2 &: H(T) \to H(T) \\ [G_2(\psi)](t) &= g_2(t)\, \psi(t) \end{aligned} \tag{5.36}$$

The dynamic matrix operator V_{12} can then be expressed in terms of G and V_0 as

$$V_{12} = GV_0G = \begin{pmatrix} 0 & U_{12}^* \\ U_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & G_1 U_0^* G_2 \\ G_2 U_0 G_1 & 0 \end{pmatrix} \tag{5.37}$$

We now suppose that the reference dynamics u_0 satisfies a space-time symmetry which leads to an order-two degeneracy of the eigenspaces of U_0. The generalization to a degeneracy of higher order does not present any additional difficulty, due to the cyclicity of $\Gamma(U_0)$. The topos and chronos of U_0 are written as $\varphi_n^{0^+}$, $\varphi_n^{0^-}$ and $\psi_n^{0^+}$, $\psi_n^{0^-}$, $n = 1, 2,\ldots$. We can thus choose a family of pairs of operators $(S_n, \tilde{S}_n) \in \Gamma(U_0)$, $n = 1, 2,\ldots$, such that

$$\begin{aligned} S_n \phi_n^{0^+} &= \phi_n^{0^-} \\ S_n \phi_n^{0^-} &= -\phi_n^{0^+} \\ S_n \psi_n^{0^+} &= \psi_n^{0^-} \\ S_n \psi_n^{0^-} &= \psi_n^{0^+} \end{aligned} \tag{5.38}$$

If G_1 and G_2 are invertible operators, as is the case if $g_i(x) \neq 0$ for all x and $g_2(t) \neq 0$ for all t, we can introduce the inverse operators, G_1^{-1} and G_2^{-1}. We then define the operators T_n and $\tilde{T}_n$ defined on $H(X)$ and $H(T)$, respectively, such that

$$\begin{aligned} T_n &= G_1^{-1} S_n G_1 \\ \tilde{T}_n &= G_2 \tilde{S}_n G_2^{-1} \end{aligned} \tag{5.39}$$

and the operator

$$T = \sum_n P_n T_n P_n \tag{5.40}$$

P_n being the matrix operator

$$P_n = \begin{pmatrix} P_n & 0 \\ 0 & \tilde{P}_n \end{pmatrix}$$

where $(P_n, \tilde{P}_n)$ are the resolutions of the identity in the characteristic spaces $\chi_0(X)$ and $\chi_0(T)$ which perform the spectral decomposition of U_0. The operator V_{12} always intertwines T and the adjoint of its inverse $(T^{-1})^*$, that is

$$(T^{-1})^* V_{12} = V_{12} T \tag{5.41}$$

Under the condition that the operators G^2 and V_0 commute on $\chi_0(X) \oplus \chi_0(T)$, i.e., $[V_0 G^2 V_0, V_0] = 0$, where $[A, B]$ denotes the commutator of A and B,[28] T is unitary and becomes a spatiotemporal symmetry for the modulated dynamics. Then, we can write

$$T V_{12} = V_{12} T \tag{5.42}$$

It is clear that, even if u_0 satisfies a point symmetry, this is not the case for u_{12}. Although the previous results are valid *independently of the complexity and the space-time symmetry of* $u_0(x, t)$,[28] we now illustrate them with a spatially and temporally modulated uniformly traveling wave for which

$$u_0(x, t) = g_0(x - ct)$$

In this case, it is convenient to consider the Fourier decomposition of u_{12}:

$$u_{12}(x, t) = \sum_{k,l,m} g_0^k g_1^l g_2^m e^{2i\pi(Mk + l)x/X} e^{-2i\pi(Nk + m)t/T} \tag{5.43}$$

where we have assumed that we can implement discrete Fourier series, which is the case, for instance, if u_{12} is defined on a space-time domain $X \times T$ where X and T are finite intervals commensurable with the speed c of the wave ($N|X| = cM|T|$, N and M being integers), and g_1 and g_2 are periodic of periods $|X|$ and $|T|$. We denote K, L and M the sets of integers k, l, and m for which the corresponding Fourier coefficients of g_0, g_1, and g_2, respectively, are nonzero. In this case, the commutation of G^2 and V_0 on $\chi_0(X) \oplus \chi_0(T)$ can be written as the following nonresonant conditions of the spatial and temporal Fourier sidebands: $\forall k, k' \in K$, $\forall l, l' \in L$,

$$Mk + l = Mk' + l' \Leftrightarrow k = k', \quad l = l' \tag{5.44}$$

and $\forall k, k' \in K$, $\forall m, m' \in M$,

$$Nk + m = Nk' + m' \Leftrightarrow k = k', \quad m = m' \tag{5.45}$$

Under the latter conditions (5.44) and (5.45), the matrix operator T

$$T = \sum_k P_k T(x_k, t_k) P_k \tag{5.46}$$

involving the translation symmetry operators S_k and $\tilde{S}_k$:

$$S_k = S_{x_k}, \qquad \text{with} \quad x_k = \frac{1}{4Nk}$$

$$\tilde{S}_k = \tilde{S}_{t_k}, \qquad \text{with} \quad t_k = \frac{1}{4Mk}$$

defines a space-time symmetry for u_{12}. Here, the biorthogonal index n has been replaced by the Fourier index k since there is always a set of topos and chronos of u_0 which are Fourier modes (since u_0 is a traveling wave) and, under the conditions (5.44) and (5.45), the modulated Fourier modes are topos and chronos of u_{12}.[28] It is clear that T is not a point symmetry.

All the previous points remain valid if g_0 is a nonperiodic function considered on the infinite domain $X \times T = R \times R$. In this case, u_{12} is also a nonperiodic function in space and time. A particular explicit example is given by the function

$$u_0(x, t) = \cos[\alpha(x - ct)] + 2\cos[\beta(x - ct)], \qquad \text{with} \quad \alpha/\beta \notin Q$$

subject to the modulations $g_1(x) = \cos(x)$ and $g_2(t) = \cos(ct)$. The nonresonance conditions (5.44) and (5.45) are satisfied if $|\alpha - \beta| > 1$, in which case the modulated wave $u_{12}(x, t)$ satisfies the space-time symmetry (5.46). (We recall that the technical treatment of the unbounded domain can be found in ref. 18.)

As in the case of two stationary waves propagating at two different uniform speeds (Example 5.4a), the spatiotemporal symmetries of two spatially and temporally modulated uniformly traveling waves, one propagating at speed c and the other one propagating at speed $2c$, are different, while the symmetries of the spatial and temporal two-point correlations are the same.

All the previous results still hold if the modulation depends on the eigenspace considered, namely the modulation operators become G_{1_n} and G_{2_n}, which then appear in the expression of the space-time symmetry (5.39). In the case of a traveling wave, this amounts to considering modulations which are wavenumber and frequency dependent.

Modulated traveling waves in thin films flowing on an incline surface were discovered via the biorthogonal analysis of experimental data.[46] The regime studied there displayed rather complex space-time dynamics exhibiting splitting and coalescence of wavefronts.

Example 5.5. We now present two functions which do not satisfy the same spatiotemporal symmetry, for which statistical symmetries are also different, but for which the temporal averages are both invariant under the same spatial symmetry. For this, we consider the traveling wave of speed c_3:

$$u_3(x, t) = g_3(x - c_3 t) \tag{5.47}$$

and the superposition of two traveling waves of speeds c_3 and c_4:

$$u_4(x, t) = g_3(x - c_3 t) + g_4(x - c_4 t) \tag{5.48}$$

While the traveling wave (5.47) is invariant under a spatiotemporal translation symmetry, this symmetry is broken in the superposition (5.48), due to the presence of space-time resonances between the two waves.[27] For the same reason, the statistical spatial and temporal symmetries are also broken. However, the time (resp. space) average of the superposition is still invariant under spatial (resp. temporal) translations.

Example 5.6. Finally, we illustrate the fact that two different functions may have the same spatial and temporal two-point correlations, due to a simple rotation of the eigenvectors inside the same eigenspace. For this, we consider

$$u_5(x, t) = A \sin(ax + bt) \tag{5.49}$$

and

$$u_6(x, t) = A \cos(ax - bt) \tag{5.50}$$

Noting that $u_5(x, t)$ can also be written as

$$u_5(x, t) = A(\cos ax \sin bt + \sin ax \cos bt)$$

and $u_6(x, t)$ as

$$u_6(x, t) = A(\sin ax \sin bt + \cos ax \cos bt)$$

we can easily see that $u_5(x, t)$ and $u_6(x, t)$ have the same temporal and spatial correlations, i.e., in terms of the corresponding operators

$$U_5 U_5^* = U_6 U_6^*$$
$$U_5^* U_5 = U_6^* U_6$$

but, obviously

$$U_5 \neq U_6$$

6. CONCLUDING REMARKS

In general, the symmetries of a dynamical system are of primary importance. While this importance has been extensively exploited for temporal systems, it has been much less investigated for spatiotemporal dynamics, although some recent (experimental) observations have drawn attention to the subject.[12–14] While the latter observations are mostly statistical (as they have been for over a century in turbulence), we have shown that there is a connection with the existence of spatiotemporal, deterministic symmetries (as introduced in ref. 1), the examples of which are numerous in physics, as soon as solutions become unsteady. With these motivations in mind, we have proposed a method to compute all the space-time symmetries of a given spatiotemporal function. These symmetries form a group, $\Gamma(U)$, which is fully determined by the dimensions $(d_1,..., d_k)$ of the spatial and temporal eigenspaces $(E_1, \tilde{E}_1),..., (E_k, \tilde{E}_k)$ of the biorthogonal decomposition of $u(x, t)$. It follows that symmetry-increasing and -decreasing bifurcations (through which the groups of symmetries before and after the bifurcation are not isomorphic) can only occur through splittings or crossings of (biorthogonal) eigenvalues as the parameter varies. Conversely we have presented a procedure for determining all the spatiotemporal functions satisfying a given space-time symmetry group. This is important from a practical viewpoint when the (symmetry) group is known *a priori.*

As far as the relations between various symmetries is concerned, we have shown in this paper that the symmetry of the time orbit in the characteristic space $\chi(X)$ is equivalent to the existence of a spatiotemporal symmetry in physical space whose temporal part is implemented on the time variable (point symmetry). A similar remark applies to the space orbit in the characteristic space $\chi(T)$. Not only is this connection between the dynamics in phase space and those in physical space essential, but it also emphasizes once again the crucial role played by spatiotemporal symmetries. Nevertheless, it presents a challenge to experimentalists since the detection of such symmetries in signals requires a complete set of simultaneous time series at multiple space positions. A reasonable insight can be reached by the analysis of the spatial or temporal two-point correlation, since the presence of a symmetry in either one is equivalent to the presence of a spatiotemporal symmetry. The full determination of the latter cannot be reached, however, by the analysis of such statistics alone, as the space-time isomorphism is then lost.

Finally, we should mention that the extension of the notion of "global" space-time symmetries (such as the translation symmetries characteristic of traveling waves) to symmetries which are "local" in the space-time domain is in progress. A step in this direction when spatial and temporal modulational instabilities occur is reported in ref. 28 (see Example 5.4b). Under certain conditions,[28] a global space-time symmetry still exists, but it is no longer pointwise.

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